

Elementary Matrix Reduction Over J-Stable Rings

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Abstract: A commutative ring R is J-stable provided that R/aR has stable range 1 for all $a \notin J(R)$. A commutative ring R in which every finitely generated ideal is principal is called a Bézout ring. A ring R is an elementary divisor ring provided that every matrix over R admits a diagonal reduction. We prove that a J-stable ring is a Bézout ring if and only if it is an elementary divisor ring. Further, we prove that every J-stable ring is strongly completable. Various types of J-stable rings are provided. Many known results are thereby generalized to much wider class of rings, e.g. [2, Theorem 8], [4, Theorem 4.1], [7, Theorem 3.7], [8, Theorem], [9, Theorem 2.1], [13, Theorem 1] and [17, Theorem 7].

Keywords: Elementary divisor ring, Bézout ring, J-Stable ring, Adequate condition.

MR(2010) Subject Classification: 13F99, 13E15, 06F20.

1. INTRODUCTION

Throughout this paper, all rings are commutative with an identity. A matrix A (not necessarily square) over a ring R admits diagonal reduction if there exist invertible matrices P and Q such that PAQ is a diagonal matrix (d_{ij}) , for which d_{ii} is a divisor of $d_{(i+1)(i+1)}$ for each i . A ring R is called an elementary divisor ring provided that every matrix over R admits a diagonal reduction. A ring is a Bézout ring if every finitely generated ideal is

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principal. Evidently, every elementary divisor ring is a Bézout ring. It is attractive to explore the conditions under which a Bézout ring (maybe with zero divisors) is an elementary divisor ring.

A ring R is a Hermite ring if every 1×2 matrix over R admits a diagonal reduction. As is well known, a ring R is Hermite if and only if for all $a, b \in R$ there exist $a_1, b_1 \in R$ such that $a = a_1d, b = b_1d$ and $a_1R + b_1R = R$ ([14, Theorem 1.2.5]). Thus, every Hermite ring is a Bézout ring. After Kaplansky's work on elementary divisor rings without nonzero zero divisors, Gillman and Henriksen proved that

Theorem 1.1 [7, Theorem 1.1]. *A ring R is an elementary divisor ring if and only if*

- (1) *R is a Hermite ring;*
- (2) *For all $a_1, a_2, a_3 \in R$, $a_1R + a_2R + a_3R = R \implies$ there exist $p, q \in R$ such that $pa_1R + (pa_2 + qa_3)R = R$.*

A ring R is said to have stable range 1 provided that $aR + bR = R$ with $a, b \in R$ implies that $a + by \in R$ is invertible for a $y \in R$. It was first introduced so as to study stabilization in algebraic K -theory. Afterwards, it was studied to deal with the cancellation problem of modules [1]. As is well known, a regular ring R is unit-regular if and only if R has stable range 1. In [3, Theorem 3], it was proved that every unit-regular ring is an elementary divisor ring. Following McGovern, a ring R has almost stable range 1 provided that every proper homomorphic image of R has stable range 1. Evidently, every ring having stable range 1 has almost stable range 1. Moreover, every neat ring (including FGC-domains and h-local domains) has almost stable range 1 (cf. [6]). In [7, Theorem 3.7], it is proved that every Bézout ring having almost stable range 1 is an elementary divisor ring.

In this article, we generalize almost stable rang 1 and introduce J-stable rings. We say that a ring R is J-stable provided that R/aR has stable range 1 for all $a \notin J(R)$. Here, $J(R)$ denote the Jacobson radical of R . Clearly, every ring having almost stable 1 is J-stable. In Section 2, we shall investigate elementary properties of J-stable rings. Various types of J-stable rings which do not have almost stable range 1 are provided.

An element $a \in R$ is adequate if for any $b \in R$ there exist some $r, s \in R$ such that (1) $a = rs$; (2) $rR + bR = R$; (3) $s'R + bR \neq R$ for each non-invertible divisor s' of s . A Bézout ring in which every nonzero element is adequate is called an adequate ring. Kaplansky proved that for the class of adequate domains being a Hermite ring was equivalent to being an elementary divisor ring. This was extended to rings with zero-divisors by Gillman and Henriksen [2, Theorem 8]. In Section 3, we consider certain subclasses of J-stable rings by means of adequate property. A Bézout ring in which every element not in $J(R)$ is adequate is called J-adequate. Clearly, every adequate ring is J-adequate. For instances, regular rings and valuation rings. A Bézout ring R is π -adequate provided that for any $a \neq 0$ there exists some $n \in \mathbb{N}$ such that $a^n \in R$ is adequate. We shall prove that every J-adequate ring and every π -adequate ring are J-stable rings, and then enrich the supply of such new rings by means of generalizations of adequate rings.

Finally, we prove, in Section 4, that every J-stable ring is a Bézout ring if and only if it is an elementary divisor ring. This gives a nontrivial generalization of [7, Theorem 3.7]. The technique here inspires us to introduce quasi adequate rings, and prove that every quasi adequate ring is an elementary divisor ring. This extend [13, Theorem 1] as well. Furthermore, we generalize [9, Theorem 2.1] and prove that every J-stable ring is strongly completeable. This also extend [8, Theorem] to much wider class of rings (maybe with zero divisors).

2. J -Stable Rings

The purpose of this section is to investigate elementary properties of J -stable rings.

Theorem 2.1. *Let R be a ring. Then the following are equivalent:*

- (1) R is J -stable.
- (2) $a_1R + a_2R + a_3R = R$ with $a_1 \notin J(R) \implies$ there exists $b \in R$ such that $a_1R + (a_2 + a_3b)R = R$.
- (3) For any $a \notin J(R)$, R/a^nR has stable range 1 for some $n \in \mathbb{N}$.

Proof. (1) \Rightarrow (3) This is obvious.

(3) \Rightarrow (2) Suppose that $aR + bR + cR = R$ with $a \notin J(R), b, c \in R$. Then $a^nR + bR + cR = R$. By hypothesis, R/a^nR has stable range 1. Clearly, $\overline{b}(R/a^nR) + \overline{c}(R/a^nR) = \overline{R/a^nR}$. Then we can find some $y \in R$ such that $\overline{b + cy} \in R/a^nR$ is invertible. Hence, $\overline{(b + cy)d} = \overline{1}$, and then $a^nx + (b + cy)d = 1$ for some $x \in R$. Therefore $aR + (b + cy)R = R$, as desired.

(2) \Rightarrow (1) Let $a \notin J(R)$. Given $\overline{bc} + \overline{d} = \overline{1}$ in R/aR , then $ax + bc + d = 1$ for some $x \in R$. By hypothesis, there exists a $y \in R$ such that $aR + (b + dy)R = R$. Hence, $\overline{b + dy}(R/aR) = R/aR$, and so $\overline{b + dy} \in R/aR$ is invertible. Therefore, R/aR has stable range 1, as desired. \square

Corollary 2.2. *Let R be a ring. then the following are equivalent:*

- (1) R is J -stable.
- (2) For every three elements $a, b, c \in R$ such that $a \notin J(R)$ and $bR + cR = R$, there exists $a y \in R$ such that $aR + (b + cy)R = R$.

Proof. (1) \Rightarrow (2) This is obvious.

(2) \Rightarrow (1) Let $a \notin J(R)$. Suppose that $\overline{b}(R/aR) + \overline{c}(R/aR) = \overline{R/aR}$ with $b, c \in R$. Write $ax + by + cz = 1$ for some $x, y, z \in R$. Hence, $by + (cz + ax) = 1$. By hypothesis, $aR + (b + (cz + ax)t)R = R$ for a $t \in R$. Thus, $aR + (b + czt)R = R$, and so $\overline{b + czt} \in R/aR$ is invertible. This implies that R/aR has stable range 1. Therefore R is J -stable, by Theorem 2.1. \square

Corollary 2.3. *Let e be an idempotent of a J -stable ring R , then eRe is J -stable.*

Proof. For every three elements $a, b, c \in eRe$ such that $a \notin J(eRe)$ and $b(eRe) + c(eRe) = eRe$, we see that $a \notin J(R)$ and $(b + 1 - e)R + cR = R$. Since R is J -stable, it follows by Corollary 2.2 that $aR + (b + 1 - e + cy)R = R$ for a $y \in R$. Write $ax + (b + 1 - e + cy)z = 1$ for some $x, z \in R$. Then $(1 - e)ze = 0$, and so $ze = eze$. This implies that $a(eze) + (b + c(eye))(eze) = e$. Hence, $a(eRe) + (b + c(eye))(eRe) = eRe$. By using Corollary 2.2 again, eRe is J -stable. \square

Following Moore and Steger, a ring R is called a B -ring provided that $a_1R + \cdots + a_nR = R$ ($n \geq 3$) with $(a_1, \dots, a_{n-2}) \not\subseteq J(R) \implies$ there exists $b \in R$ such that $a_1R + \cdots + a_{n-2}R + (a_{n-1} + a_nb)R = R$. Elementary properties of such rings have been studied in [9]. Surprisingly, we shall prove the classes of J -stable rings and B -rings coincide with each other. That is,

Proposition 2.4. *Let R be a ring. then the following are equivalent:*

(1) R is J -stable.

(2) $a_1R + \cdots + a_nR = R$ ($n \geq 3$) with $(a_1, \dots, a_{n-2}) \not\subseteq J(R) \implies$ there exists $b \in R$ such that $a_1R + \cdots + a_{n-2}R + (a_{n-1} + a_nb)R = R$.

Proof. (1) \implies (2) The assertion is true for $n = 3$, by Theorem 2.1. Suppose the result holds for $n = k$ ($k \geq 3$). Given $a_1R + \cdots + a_{k+1}R = R$ ($n \geq 3$) with $(a_1, \dots, a_{k-1}) \not\subseteq J(R)$, then there are $x_1, \dots, x_{k+1} \in R$ such that $(a_1x_1 + a_2x_2) + a_3x_3 + \cdots + a_{k-1}x_{k-1} + a_kx_k + x_{k+1}x_{k+1} = 1$. Hence, $(a_1x_1 + a_2x_2)R + a_3R + \cdots + a_{k-1}R + a_kR + a_{k+1}R = R$. By hypothesis, $(a_1x_1 + a_2x_2)R + a_3R + \cdots + a_{k-1}R + (a_k + a_{k+1}z)R = R$ for some $z \in R$. Therefore $a_1R + a_2R + \cdots + a_{k-1}R + (a_k + a_{k+1}z)R = R$. By induction, we complete the proof.

(2) \implies (1) This is obvious. \square

By virtue of Proposition 2.4, we see that J -stable rings and B -rings coincide, but we prefer to use this new concept as the preceding Theorem 2.1 shows that it is close to stable range 1. This observation provides many class of such rings. For instances, semi-local rings, local rings, π -regular rings, regular rings, Noetherian rings in which every proper prime ideal is maximal (in particular, Dedekind domain) are all J -stable. Let $a \in R$ and $\text{mspec}(a) = \{M \in \text{Max}(R) \mid a \in M\}$. Further, we have

Example 2.5. Let R be a ring in which $\text{mspec}(a)$ is finite for all $a \notin J(R)$. Then R is J -stable.

Let R be ring in which R/aR is semilocal for all $a \notin J(R)$. Then R is J -stable. As every semilocal ring has stable range 1, we are done by Theorem 2.1.

Lemma 2.6. A ring R has stable range 1 if and only if $a_1R + a_2R + a_3R = R$ implies that $a_1R + (a_2 + a_3b)R = R$ for some $b \in R$.

Proof. \implies Suppose $a_1R + a_2R + a_3R = R$. Then $a_1x_1 + a_2x_2 + a_3x_3 = 1$; hence, $a_2R + (a_1x_1 + a_3x_3)R = R$. Thus, we have a $y \in R$ such that $a_2 + (a_1x_1 + a_3x_3)y = u \in U(R)$. Hence, $a_1x_1yu^{-1} + (a_2 + a_3x_3y)u^{-1} = 1$. Therefore, $a_1R + (a_2 + a_3b)R = R$, where $b = x_3y$.

\Leftarrow This is obvious. \square

Theorem 2.7. Let $\{R_i \mid i \in I\}$ ($|I| \geq 2$) be a family of rings. Then the direct product $R = \prod R_i$ of rings R_i is J -stable if and only if each R_i has stable range 1.

Proof. \implies Given $aR_1 + bR_1 + cR_1 = R_1$ with $a, b, c \in R$, then

$$(a, 1, 1, \dots)R + (b, 0, 0, \dots)R + (c, 0, 0, \dots)R = R.$$

Clearly, $(a, 1, 1, \dots, 1) \notin J(R)$. By hypothesis, there exists $(y, y_2, y_3, \dots) \in R$ such that

$$(a, 1, 1, \dots)R + ((b, 0, 0, \dots) + (c, 0, 0, \dots)(y, y_2, y_3, \dots))R = R.$$

Therefore, $aR_1 + (b + cy)R_1 = R_1$. In light of Lemma 2.6, R_1 has stable range 1. Likewise, R_i has stable range 1 for $i \neq 1$. Therefore each R_i has stable range 1.

\Leftarrow Since each R_i has stable range one, we see that $R = \prod R_i$ has stable range 1. Therefore, R is J -stable. \square

Corollary 2.8. Let $L = \prod_{i \in I} R_i$ be the direct product of rings $R_i \cong R$ and $|I| \geq 2$. Then L is J -stable if and only if R has stable range 1 if and only if L has stable range 1.

Thus, we see that $\mathbb{Z} \times \mathbb{Z}$ is not J-stable, while \mathbb{Z} is J-stable.

Proposition 2.9. *Let R be a ring, and let I be an ideal of R . If $I \subseteq J(R)$, then the following are equivalent:*

- (1) R is J-stable.
- (2) R/I is J-stable.

Proof. (1) \Rightarrow (2) Let R be a J-stable ring and let $\bar{a}_1\bar{R} + \bar{a}_2\bar{R} + \bar{a}_3\bar{R} = \bar{R}$, where $\bar{a}_1 \notin J(\bar{R})$, so there are $\bar{r}_1, \bar{r}_2, \bar{r}_3 \in \bar{R}$ such that $\bar{a}_1\bar{r}_1 + \bar{a}_2\bar{r}_2 + \bar{a}_3\bar{r}_3 = \bar{1}$. Hence, $a_1r_1 + a_2r_2 + a_3r_3 = 1 + x$ for some $x \in I$ as $I \subseteq J(R)$, $1 + x$ is a unit and then $(a_1r_1 + a_2r_2 + a_3r_3)R = R$, hence $a_1R + a_2R + a_3R = R$ with $a_1 \notin J(R)$. By Theorem 2.1, there exists $b \in R$ such that $a_1R + (a_2 + a_3b)R = R$ and then $\bar{a}_1\bar{R} + (\bar{a}_2 + \bar{a}_3\bar{b})\bar{R} = \bar{R}$.

(2) \Rightarrow (1) Let $a_1R + a_2R + a_3R = R$, with $a_1 \notin J(R)$, so we have $\bar{a}_1\bar{R} + \bar{a}_2\bar{R} + \bar{a}_3\bar{R} = \bar{R} = R/I$ as R/I is a J-stable ring, there exists $\bar{b} \in \bar{R}$ such that $\bar{a}_1\bar{R} + (\bar{a}_2 + \bar{a}_3\bar{b})\bar{R} = \bar{R}$ so $a_1r_1 + (a_2 + a_3b)r_2 = 1 + y$ for some $y \in I$. As $1 + y \in U(R)$, we have $a_1R + (a_2 + a_3b)R = R$. According to Theorem 2.1, R is J-stable. \square

It follows by Proposition 2.9 that R is J-stable if and only if $R/J(R)$ is J-stable. Also we see that every homomorphic image of a J-stable ring is J-stable. Thus, R is J-stable if and only if R/aR is J-stable for any $a \in R$.

Corollary 2.10. *Let R be a ring. Then the following are equivalent:*

- (1) R is J-stable.
- (2) $R[[x_1, \dots, x_n]]$ is J-stable.

Proof. Let $\psi : R[[x_1, \dots, x_n]] \rightarrow R$ is defined by $\psi(f(x_1, \dots, x_n)) = f(0, \dots, 0)$ it is easily prove that ψ is a ring epimorphism such that $\ker \psi \subseteq J(R[[x_1, \dots, x_n]])$, now the result follows from the Proposition 2.9. \square

Let R be a ring, and let M be an R - R -bimodule. Then the trivial extension $T(R, M)$ is the ring $\{(r, m) \mid r \in R, m \in M\}$, where the operations are defined as follows: For any $r_1, r_2 \in R, m_1, m_2 \in M$,

$$\begin{aligned} (r_1, m_1) + (r_2, m_2) &= (r_1 + r_2, m_2 + m_2), \\ (r_1, m_1)(r_2, m_2) &= (r_1r_2, r_1m_2 + m_1r_2). \end{aligned}$$

Corollary 2.11. *Let R be a ring, and let M be an R - R -bimodule. Then the following are equivalent:*

- (1) R is J-stable.
- (2) $T(R, M)$ is J-stable.

Proof. Let $\psi : R \rightarrow T(R, M)/J(T(R, M))$ be such that $\psi(r) = \overline{(r, 0)}$ for any $r \in R$ $\ker(\psi) = \{r \in R \mid (r, 0) \in J(T(R, M))\} = \{r \in R \mid r \in J(R)\}$. As $J(T(R, M)) = \{\overline{(r, m)} \mid r \in J(R), m \in M\}$. Also for any $(r, m) \in T(R, M)/J(T(R, M))$ we can write $(r, m) = (r, 0) + (0, m) = \psi(r) + 0 = \psi(r)$ that shows ψ is surjective, now we can get the result by Proposition 2.9. \square

As an immediate consequence, we deduce that a ring R is J-stable if and only if the ring $\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$ is J-stable.

Clearly, every ring of stable range 1 is J-stable. Further, all rings having almost stable rang 1 is J-stable. But the J-stable rings having not almost stable range 1 are rich.

Example 2.12. Let $R = \{a + bx \mid a, b \in \mathbb{Z}, x^2 = 0\}$. Then R is a J-stable ring, while it does not have almost stable range 1.

Proof. Clearly, $J(R) = \{bx \mid a, b \in \mathbb{Z}, x^2 = 0\}$. Let $z \notin J(R)$. Write $z = c + dx, c, d \in \mathbb{Z}$ and $c \neq 0$. Construct a map $\varphi : R/zR \rightarrow \mathbb{Z}/c\mathbb{Z}, \overline{a + bx} \mapsto \overline{a}$. Then $(R/zR)/\ker(\varphi) \cong \mathbb{Z}/c\mathbb{Z}$. In view of Example 2.5, $\mathbb{Z}/c\mathbb{Z}$ has stable range 1. On the other hand, $\ker(\varphi) = \{\overline{bx} \mid b \in R\} \subseteq J(R/zR)$. Thus, R/zR has stable range 1. This implies that R is J-stable. Obviously, $R/xR \cong \mathbb{Z}$ does not have stable range 1. Therefore R does not have almost stable range 1. \square

Example 2.13. Let $R = \{z_0 + a_1x + a_2x^2 + \cdots + \mid z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}\}$. Then R is a J-stable ring, while it does not have almost stable range 1. Here, $J(R) = x\mathbb{Q}[[x]]$, and then $R/J(R) \cong \mathbb{Z}$ does not have stable range 1. Thus, R does not have almost stable range 1. Let $f(x) = z_0 + a_1x + a_2x^2 + \cdots \notin J(R)$. Then $z_0 \neq 0$. Let $I = (f(x))$. Then $f(x) = z_0(1 + z_0^{-1}a_1x + z_0^{-1}a_2x^2 + \cdots) \in I$. This implies that $z_0 \in I$. For any $g(x) \in J(R)$, we see that $g(x) \in x\mathbb{Q}[[x]]$. Clearly, $z_0(1 + z_0^{-1}g(x)) \in I$, and so $z_0 + g(x) \in I$. This implies that $g(x) \in I$. Thus, $J(R) \subseteq I$. We infer that $I/J(R)$ is a proper ideal of $R/J(R)$. Since $R/I \cong R/J(R)/I/J(R)$, we see that R/I has stable range 1, therefore R is J-stable.

Example 2.14. Let R be the collection of all elements of the form $a\alpha + b\beta + c\gamma + d$, with $a, b, c, d \in \mathbb{Z}_2$, where α, β, γ satisfy the relations

$$\alpha^2 = \beta^2 = \gamma^2 = \alpha\beta = \beta\alpha = \alpha\gamma = \gamma\alpha = \beta\gamma = \gamma\beta = 0.$$

Then $R[x]$ is J-stable, but $R[x]$ does not have almost stable range 1. As R is consisted entirely of nilpotent elements $\{0, \alpha, \beta, \gamma, \alpha + \beta, \alpha + \beta, \beta + \gamma, \alpha + \beta + \gamma$ and units $1, 1 + \alpha, 1 + \beta, 1 + \gamma, 1 + \alpha + \beta, 1 + \alpha + \gamma, 1 + \beta + \gamma, 1 + \alpha + \beta + \gamma$. It follows by [11, Example 3.4] and Proposition 2.4 that $R[x]$ is J-stable. Let $I = (\alpha)[x]$. Then I is a proper ideal of $R[x]$. Since $R[x]/I \cong (R/(\alpha))[x]$ has not stable range 1. Therefore, $R[x]$ does not have almost stable range 1.

By [9, Theorem 2.7] and Proposition 2.4, we see that $R[x, y] = R[x][y]$ is not J-stable for an arbitrary ring R .

Example 2.15. Let K be a field and x be an undeterminate on K , and let $R = K[x]$. As R is a principal ideal domain, it has almost stable range 1, which does not have stable range 1. So it is J-stable by Theorem 2.1. It follows from Corollary 2.10 that $R[[y]] = K[x][[y]]$ is J-stable. Now we have $J(R[[y]]) = (y, J(R))$ which is non-zero ideal of $R[[y]]$. Clearly, $R[[y]]/J(R[[y]]) \cong R/J(R)$. As $R/J(R)$ does not have stable range 1, and so $R[[y]]/J(R[[y]])$ does not have stable rang 1. Therefore it shows that $R[[y]]$ does not have almost stable range 1, and then we are through.

3. Certain Subclasses

An element $a \in R$ is clean if it is the sum of an idempotent and a unit. A ring R is clean provided that every element in R is clean. By virtue of [1, Theorem 17.2.2], every clean ring has stable range 1.

Lemma 3.1 [15, Theorem 2 and Theorem 4]. *Let R be a Bézout ring. If $a \in R$ is adequate, then R/aR is clean.*

Theorem 3.2. *Every J -adequate ring is J -stable.*

Proof. Let R be J -adequate, and let $a \notin J(R)$. By hypothesis, $a \in R$ is adequate. In view of Lemma 3.1, R/aR is clean, and so R/aR has stable range 1. Therefore R is J -stable, as asserted. \square

Recall that a ring R is called an NJ-ring if every element $a \notin J(R)$ is regular [10]. For instance, every regular ring and every local ring are NJ-rings.

Example 3.3. *Every Bézout NJ-ring is J -adequate.*

Proof. Let R be a Bézout NJ-ring. In view of [10, Theorem 2], R must be a regular ring, a local ring or isomorphic to the ring of a Morita context with zero pairings where the underlying rings are both division ring. If R is regular, then it is adequate. If R is local, it is J -adequate. If R is isomorphic to the ring of a Morita context $T = (A, B, M, N, \varphi, \phi)$ with zero pairings, where the underlying rings are division rings A and B . Then R is not commutative, a contradiction. Therefore R is J -adequate. \square

It should be noted that adequate rings were studied in many papers (see [15]). Obviously, every adequate ring is J -adequate. But the converse is not true, as the following shows.

Example 3.4. *Let $R = \{z_0 + a_1x + a_2x^2 + \cdots \mid z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}\}$. Then R is a J -adequate ring, while it is not an adequate ring.*

Proof. As in [12, Example 3.3], R is a Bézout domain, but it is not an adequate ring. Let $f(x) = y + b_1x + b_2x^2 + \cdots \notin J(R)$ and $g(x) = z + c_1x + c_2x^2 + \cdots \in R$. Then $y \neq 0$. Since \mathbb{Z} is a principal ideal domain, it is adequate. Thus, there exist $s, t \in R$ such that $y = st$, $(s, z) = 1$, and that $(t', z) \neq 1$ for any non-unit divisor t' of t . If $(s, t) \neq 1$, then we have a nonunit $d \in R$ such that $(s, t) = d$. Hence, $(d, z) \neq 1$, and then $(s, z) \neq 1$, an absurd. Therefore $(s, t) = 1$, and so $f(x) = (s + d_1x + d_2x^2 + \cdots)(t + e_1x + e_2x^2 + \cdots)$, where d_i and e_i are solutions of the equations:

$$\begin{aligned} se_1 + d_1t &= b_1; \\ se_2 + d_2t &= b_2 - d_1e_1; \\ se_3 + d_3t &= b_3 - d_1e_2 - d_2e_1; \\ &\vdots \end{aligned}$$

Set $s(x) = s + d_1x + d_2x^2 + \cdots$ and $t(x) = t + e_1x + e_2x^2 + \cdots$. Clearly, we can find some $k, l \in \mathbb{Z}$ such that $ks + lz = 1$. Hence, $1 - (ks(x) + lg(x)) \in J(R)$. Thus, $ks(x) + lg(x) \in U(R)$. This shows that $(s(x), g(x)) = 1$. If $t'(x) = m + f_1x + f_2x^2 + \cdots$ is a nonunit divisor of $t(x)$, then m is a nonunit divisor of t . By hypothesis, $(m, z) \neq 1$. This implies that $(t'(x), g(x)) \neq 1$. Therefore R is J -adequate, as asserted. \square

Lemma 3.5. *Let R be a J -adequate ring. Then every prime ideal not in $J(R)$ contains in a unique maximal ideal of R .*

Proof. Let $P \not\subseteq J(R)$ be a prime ideal of R . Then P contains at least one adequate element. As in the proof of [14, Proposition 3.2.4], P is contained in a unique maximal ideal of R . \square

Let R be a J -adequate ring. Then R/P is a local ring for all prime ideal $P \not\subseteq J(R)$. Recall that a ring is a pm ring provided that every prime ideal is contained in a unique maximal ideal of R (see [5]). As is well known, a ring R is a pm ring if and only if $a + b = 1$ implies that $(1 - ax)(1 - by) = 0$ for some $x, y \in R$. Further, we derive

Theorem 3.6. *Let R be a J -adequate ring. Then $R/J(R)$ is a pm ring.*

Proof. Let $a, b \in R$ be such that $a + b = 1$. Set $A = \{1 + ax \mid x \in R\}$, $B = \{1 + bx \mid x \in R\}$ and $T = AB$. We claim that $J(R) \cap T \neq \emptyset$. If not, we have a nonempty set $\Omega = \{I \triangleleft R \mid I \cap T = \emptyset\}$, as $J(R) \in \Omega$. Given $I_1 \subseteq I_2 \subseteq \dots$ in Ω , then $I_i \cap T = \emptyset$. Hence, $(\bigcup_i I_i) \cap T = \bigcup_i (I_i \cap T) = \emptyset$, and so $\bigcup_i I_i \in \Omega$. Thus, Ω is inductive. By Zorn's Lemma, there exists an ideal P of R , which is maximal in Ω . If $P \notin \text{Spec}(R)$, there exist $c, d \in R$ such that $c, d \notin P$, while $cd \in P$. Then $(RcR + P) \cap T, (RdR + P) \cap T \neq \emptyset$. It follows that $((RcR + P)(RdR + P)) \cap T \neq \emptyset$, and so $(RcdR + P) \cap T \neq \emptyset$, a contradiction. Therefore $P \in \text{Spec}(R)$.

If $RaR + P = R$, then $az + p = 1$ for some $z \in R, p \in P$. This implies that $p = 1 - az \in A \subseteq T$, an absurd. Thus, $RaR + P \subseteq M$ for an $M \in \text{Max}(R)$. Likewise, $RbR + P \subseteq N$ for an $N \in \text{Max}(R)$. Hence, $P \subseteq M \cap N$. But $M \neq N$; otherwise, $1 = a + b \in M = N$. By virtue of Lemma 3.5, $P \subseteq J(R)$, and so $1 - az \in J(R)$. It follows that $J(R) \cap T \neq \emptyset$, a contradiction. Accordingly, $J(R) \cap (AB) \neq \emptyset$. Therefore $(1 + ar)(1 + bs) \in J(R)$ for some $r, s \in R$.

Given $\overline{a+b} = \overline{1}$ in $R/J(R)$, then $ax + by = 1$ for some $x, y \in R$. By the preceding discussion, $(1 + axr)(1 + bys) \in J(R)$ for some $r, s \in R$. Hence, $\overline{(1 + axr)(1 + bys)} = \overline{0}$. Therefore $R/J(R)$ is a pm ring, as asserted. \square

Corollary 3.7. *Let R be a J -adequate ring. Then the following are equivalent:*

- (1) $\text{Max}(R)$ is zero-dimensional.
- (2) $\text{Max}(R/J(R))$ is zero-dimensional.

Proof. Since R is J -adequate, $R/J(R)$ is a pm ring by Theorem 3.6. In light of [1, Corollary 17.1.14], $R/J(R)$ is clean if and only if $\text{Max}(R/J(R))$ is zero-dimensional. We note that $\text{Max}(R)$ is zero-dimensional if and only if $R/J(R)$ is clean, and therefore the result follows. \square

Our next aim is to explore π -adequate rings. This will provide a new type of J -stable rings. We are now ready to prove:

Theorem 3.8. *Every π -adequate ring is J -stable.*

Proof. Let R be a π -adequate ring, and let $a \neq 0$. Then $a^n \in R$ is adequate for some $n \in \mathbb{N}$. In view of Lemma 3.1, $R/a^n R$ is clean. By virtue of [1, Theorem 17.2.2], $R/a^n R$ has stable range 1. Therefore R is J -stable, by Theorem 2.1. \square

We now study the possible transfer of the π -adequate property to homomorphic images and direct products.

Proposition 3.9. *If R is a π -adequate ring and I is an ideal of R contained in $J(R)$, then R/I is a π -adequate ring.*

Proof. Let $\bar{a}, \bar{b} \in R/I$ with $\bar{a} \neq \bar{0}$. Then $a, b \in R$ and $a \neq 0$. By hypothesis, there exists some $n \in \mathbb{N}$ such that $a^n \in R$ is adequate. Thus, $a^n = rs$ and $(r, b) = 1$. Hence, $\bar{a}^n = \bar{r}\bar{s}$ and $(\bar{r}, \bar{b}) = \bar{1}$. Let \bar{s}' be a nonunit in R/I which divides \bar{s} . Then s' is a nonunit in R . Further, s' divides $s + k$ for some $k \in J(R)$. Thus, $(s', s) \neq 1$. Since R is Bézout, write $(s', s) = (u)$. Clearly, $u \in R$ is a nonunit. As $u \mid s$, we see that $(u, b) \neq 1$. Hence, $(\bar{u}, \bar{b}) \neq 1$. It follows from $\bar{u} \mid \bar{s}'$ that $(\bar{s}', \bar{b}) \neq 1$. This completes the proof. \square

Lemma 3.10. *Let $a \in R$ be adequate. Then $a^n \in R$ is adequate for all $n \in \mathbb{N}$.*

Proof. This is obvious as in the proof of [14, Proposition 3.2.2]. \square

Theorem 3.11. *Let $\{R_i \mid i \in I\}$ ($2 \leq |I| < \infty$) be a family of rings. Then the direct product $R = \prod R_i$ of rings R_i is π -adequate if and only if*

- (1) *each R_i is π -adequate;*
- (2) *$0 \in R_i$ is adequate for all $i \in I$.*

Proof. \implies (1) Let $a_1, b_1 \in R_1$. Then $(a_1, 1, 0, \dots, 0), (b_1, 1, 0, \dots, 0) \in R$. By hypothesis, $(a_1, 1, 0, \dots, 0)^n \in R$ is adequate for some $n \in \mathbb{N}$. Hence, there exists some $(r_1, r_2, r_3, \dots, r_m), (s_1, s_2, s_3, \dots, s_m) \in R$ such that $(a_1, 1, 0, \dots, 0)^n = (r_1, r_2, r_3, \dots, r_m), (s_1, s_2, s_3, \dots, s_m)$ and $((r_1, r_2, r_3, \dots, r_m), (b_1, 1, 0, \dots, 0)) = (1, 1, 1, \dots, 1)$. Then we have $a_1^n = r_1 s_1$ and $(r_1, b_1) = 1$. Now suppose that s'_1 be a nonunit divisor of s_1 , then $(s'_1, 1, 1, \dots, 1)$ is a nonunit divisor of $(s_1, s_2, s_3, \dots, s_m)$. Now assume that $(s'_1, b_1) = 1$ then $((s'_1, 1, 1, \dots, 1), (b_1, 1, 0, \dots, 0)) = (1, 1, 1, \dots, 1)$, which is a contradiction. Thus, R_1 is a π -adequate ring. Likewise, each $R_i (i \geq 2)$ is π -adequate.

(2) Choose $(0, 1, 0, \dots, 0) \in R$. Let $b_1 \in R$. Then there exists some $(r_1, r_2, r_3, \dots, r_m), (s_1, s_2, s_3, \dots, s_m) \in R$ such that $(0, 1, 0, \dots, 0)^n = (r_1, r_2, r_3, \dots, r_m), (s_1, s_2, s_3, \dots, s_m)$ and $((r_1, r_2, r_3, \dots, r_m), (b_1, 1, 0, \dots, 0)) = (1, 1, 1, \dots, 1)$. Hence, $0 = r_1 s_1, (r_1, b_1) = 1$. If s'_1 is a nonunit divisor of s_1 . Then $(s'_1, 1, \dots, 1)$ is a nonunit divisor of $(s_1, s_2, s_3, \dots, s_m)$. By hypothesis, $((s'_1, 1, \dots, 1), (b_1, 1, 0, \dots, 0)) \neq (1, 1, \dots, 1)$. It follows that $(s'_1, b_1) \neq 1$. Therefore $0 \in R_1$ is adequate. Similarly, $e_i \in R_i (i \geq 2)$ is adequate.

\Leftarrow Let $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in R$. By hypothesis, every element in R_i is π -adequate. Then for each $1 \leq i \leq n, a_i \in R_i$ is π -adequate, so there exist $n_i \in \mathbb{N}$ such that $a_i^{n_i}$ is an adequate element of R_i . Set $m = \prod_{i=1}^n n_i$. By virtue of Lemma 3.10, $a_i^m \in R_i$ is an adequate element for each $1 \leq i \leq n$. Thus, there are $r_i, s_i \in R_i$ such that $a_i^m = r_i s_i$ and $(r_i, b_i) = 1$ and for each nonunit divisor s'_i of s_i , $(s'_i, b_i) \neq 1$. Then $(r_1, r_2, \dots, r_n)(s_1, s_2, \dots, s_n) = (a_1^m, a_2^m, \dots, a_n^m) = (a_1, a_2, \dots, a_n)^m$. Now let $(s'_1, s'_2, \dots, s'_n)$ be a nonunit divisor of (s_1, s_2, \dots, s_n) , then there exists some s'_j which is a nonunit divisor of s_j . Thus, $(s'_j, b_j) \neq 1$. This implies that $((s'_1, s'_2, \dots, s'_n), (b_1, b_2, \dots, b_n)) \neq 1$. Therefore R is an adequate ring. \square

Let R be a domain. We note that R is a π -adequate ring if and only if R is an adequate ring. One direction is obvious. As in the proof of [14, Proposition 3.2.3], every divisor of an adequate element in a Bézout domain is adequate, and then the converse is true.

As is well known, every regular ring is adequate. Furthermore, we derive

Theorem 3.12. *Every Bézout ring in which every prime ideal is maximal is π -adequate.*

Proof. Let R be a Bézout ring in which every prime ideal is maximal. Then R is π -regular. Let $a \in R$. Then we have some $m \in \mathbb{N}$ such that $a^m \in R$ is unit-regular. We claim that a^m is adequate. Let $b \in R$ be an arbitrary element so there exists some $n \in \mathbb{N}$ such that b^n is also unit regular, then there are invertible elements, $u, v \in R$ such that $a^m = a^m u a^m$ and $b^n = b^n v b^n$. Set $e = a^m u$ and $f = b^n v$ then e, f are idempotents. Now define $r = e + f - ef$. We see that $(r) = (e, f)$, and let $s = 1 - f + ef$ then $e = sr$ and $(s, f) = 1$. Thus, $a^m = s(ru^{-1})$. Furthermore, $(s, b^n) = 1$, and so $(s, b) = 1$. Since r divides f , for every non-invertible divisor x of ru^{-1} , we get $(x, f) \neq 1$. This implies that $(x, b^n) \neq 1$, and so $(x, b) \neq 1$. Therefore $a^m \in R$ is adequate, as required. \square

Corollary 3.13. *Every finite Bézout ring is π -adequate.*

Proof. Since R is finite, it is π -regular, and then every prime ideal of R is maximal. In light of Theorem 3.12, we complete the proof. \square

4. Matrices over J-Stable Rings

A ring R has stable range 2 provided that $a_1 R + a_2 R + a_3 R = R \implies$ there exist $y_1, y_2 \in R$ such that $(a_1 + a_3 y_1)R + (a_2 + a_3 y_2)R = R$.

Proposition 4.1. *Every J-stable ring has stable range 2.*

Proof. Let R be J-stable. Suppose that $a_1 R + a_2 R + a_3 R = R$ with $a_1, a_2, a_3 \in R$. If $a_1 \notin J(R)$, then there exists $b \in R$ such that $a_1 R + (a_2 + a_3 b)R = R$. If $a_1 \in J(R)$, then there exist $x_1, x_2, x_3 \in R$ such that $a_2 x_2 + a_3 x_3 = 1 - a_1 x_1 \in U(R)$. Hence, $a_2 x_2 (1 - a_1 x_1)^{-1} + a_3 x_3 (1 - a_1 x_1)^{-1} = 1$. Thus,

$$a_1 + a_3 x_3 (1 - a_1 x_1)^{-1} + a_2 x_2 (1 - a_1 x_1)^{-1} = 1 + a_1 \in U(R).$$

It follows that

$$(a_1 + a_3 x_3 (1 - a_1 x_1)^{-1})(1 + a_1)^{-1} + (a_2 + a_3 \cdot 0)(1 + a_1)^{-1} = 1.$$

Hence, $(a_1 + a_3 x_3 (1 - a_1 x_1)^{-1})R + (a_2 + a_3 \cdot 0)R = R$, and therefore R has stable range 2. \square

But the converse of Proposition 4.1 is not true. For instance, $\mathbb{Z}_6[x]$ has stable range 2, but it is not J-stable, as $\mathbb{Z}_6[x]/(2) \cong \mathbb{Z}_2[x]$ has not stable range 1. Thus, we see that $\{\text{rings having almost stable range 1}\} \subsetneq \{\text{J-stable rings}\} \subsetneq \{\text{rings having stable range 2}\}$. A ring R is called completable provided that $a_1 R + \cdots + a_n R = R, a_i \in R, i = 2, \dots, n$, implies there is a matrix over R with first row a_1, \dots, a_n and $\det(A) = 1$. As is well known, a ring R is completable if and only if every stable free R -module P , i.e. $P \oplus R^m \cong R^n$ for some $m, n \in \mathbb{N}$, is free. Since every ring having stable range 2 is completable (cf. [11, Corollary 2.1]), we see that every stable free module over J-stable rings is free. Moreover, we have

Corollary 4.2. *Let R be a J-stable ring. Then for any idempotent $e \in R, e \in a_1 R + \cdots + a_n R, a_i \in R, i = 2, \dots, n$, implies there is a matrix over R with first row a_1, \dots, a_n and $\det(A) = e$.*

Proof. Write $e = a_1x_1 + \cdots + a_nx_n$. Then $e = (ea_1)(ex_1) + \cdots + (ea_n)(ex_n)$. In view of Corollary 2.3, eRe is a J-stable ring. Thus, we can find a matrix $(a_{ij}) \in M_n(eRe)$

whose first row is (ea_1, \dots, ea_n) such that $\det(a_{ij}) = e$. Hence, we get

$$\begin{vmatrix} a_1 & \cdots & a_n \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} ea_1 & \cdots & ea_n \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = e, \text{ as required.} \quad \square$$

We turn now to the proof of our main result.

Theorem 4.3. *Let R be a J-stable ring. Then R is a Bézout ring if and only if R is an elementary divisor ring.*

Proof. \implies In view of Proposition 3.1, R has stable range 2. Thus, R is Hermite, by [7, Theorem 3.4]. Suppose that $aR + bR + cR = R$ with $a, b, c \in R$. If $a \notin J(R)$, then there exists a $z \in R$ such that $aR + (b + cz)R = R$. If $a \in J(R)$, then there exist some $x_1, x_2, x_3 \in R$ such that $ax_1 + bx_2 + cx_3 = 1$. Hence, $bx_2(1 - ax_1)^{-1} + cx_3(1 - ax_1)^{-1} = 1$. Thus,

$$(x_2(1 - ax_1)^{-1})a + bx_2(1 - ax_1)^{-1} + cx_3(1 - ax_1)^{-1} = 1 - (x_2(1 - ax_1)^{-1})a \in U(R).$$

Therefore,

$$\begin{aligned} & (x_2(1 - ax_1)^{-1})a(1 - (x_2(1 - ax_1)^{-1})a)^{-1} + (bx_2(1 - ax_1)^{-1} + cx_3(1 - ax_1)^{-1}) \\ & \quad (1 - (x_2(1 - ax_1)^{-1})a)^{-1} = 1. \end{aligned}$$

Hence, $(x_2(1 - ax_1)^{-1})aR + (x_2(1 - ax_1)^{-1}b + x_3(1 - ax_1)^{-1}c)R = R$. In light of Theorem 1.1, R is an elementary divisor ring.

\Leftarrow This is obvious, by Theorem 1.1. \square

In [7, Remark 4.7], W.W. McGovern asked a question: is there any elementary divisor domain which does not have almost stable range 1? This was affirmatively answered by Roitman in [12, Example 3.3]. In fact, J-stable Bézout domains which do not have almost stable range 1 provide rich such examples. For instance, construct R as in Example 2.13. Then R is a J-stable Bézout domain. Thus, R is an elementary divisor ring, in terms of Theorem 4.3. In this case, R does not have almost stable range 1. Further, we derive

Corollary 4.4 [7, Theorem 3.7]. *Let R have almost stable range 1. Then R is a Bézout ring if and only if R is an elementary divisor ring.*

Proof. By virtue of Theorem 2.1, R is a J-stable ring. This completes the proof, in terms of Theorem 4.3. \square

Corollary 4.5. *Every J-adequate ring is an elementary divisor ring.*

Proof. Let R be a J-adequate ring. Then R is a J-stable ring by Theorem 3.2. Therefore we complete the proof from Theorem 4.3. \square

In view of Example 3.3, every Bézout NJ-ring is J-adequate. By Corollary 4.5, we give an affirmative answer to [15, Question 4] for commutative rings.

Corollary 4.6. Every Hermite NJ-ring is an elementary divisor ring.

Recall that a ring R satisfies (N) provided that for any $a, b \in R$ and $a \notin J(R)$, there exists $m \in R$ such that $bR + mR = R$ and if some $n \in R$, $nR + aR \neq R$ and $nR + bR = R$ implies $nR + mR \neq R$. We extend [4, Corollary 2.6] as follows.

Corollary 4.7. Every Bézout ring satisfying (N) is an elementary divisor ring.

Proof. As in the note in [4, page 235] and Theorem 2.1, every Bézout ring satisfying (N) is a J-stable ring. Therefore the result follows by Theorem 4.3. \square

We say that $a \in R$ is π -adequate to $b \in R$ provided that there exists some $n \in \mathbb{N}$ such that a^n is adequate to b . A Hermite ring is called a quasi adequate ring if for any pair of nonzero elements, at least one of these elements is π -adequate to the other. It is obvious that every generalized adequate ring (including adequate ring) is quasi adequate ring (cf. [13]). We now generalize [13, Theorem 1] as follows.

Theorem 4.8. Every quasi adequate ring is an elementary divisor ring.

Proof. Let R be a quasi adequate ring. By Theorem 1.1 and [12, Theorem 2.5], it suffices to consider the matrix $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ where $aR + bR + cR = R$. If $a = 0$ then $bR + cR = R$. Hence $pb + qc = 1$ for some $p, q \in R$. Thus,

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} A \begin{pmatrix} c & p \\ b & q \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}.$$

If $c = 0$ then $aR + bR = R$, and so $pa + qb = 1$ for some $p, q \in R$. Thus,

$$\begin{pmatrix} p & q \\ -b & a \end{pmatrix} A \begin{pmatrix} 1 & -qc \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & ac \end{pmatrix}.$$

So we may assume that $a, c \neq 0$. Since R is quasi adequate ring, at least one of the elements a, c is π -adequate to the other. Let c be π -adequate to a , then there exist some $n \in \mathbb{N}$ such that $c^n = rs$, where $rR + aR = R$ and $s'R + aR \neq R$ for each non-invertible divisor s' of s . We claim that $(a + br)R + c^n rR = R$. Since R is a Hermite ring, it is a Bézout ring. If $(a + br)R + c^n rR \neq R$, then there exists a non-invertible element h of R such that $(a + br)R + c^n rR = hR$. Write $hR + rR = kR$ for some $k \in R$. Write $h = kp$ for a $p \in R$, then $a + br = kpq$ for a $q \in R$. Write $r = km$ for some $m \in R$. Then $a = k(pq - mb)$. Since $rR + aR = kmR + k(pq - mb)R = R$, we see that $kR = R$, and so $hR + rR = R$. Thus, $hu + r^2v = 1$ for some $u, v \in R$. Clearly, $c^n r = ht$ for a $t \in R$. Then $r^2s = ht$; hence, $(1 - hu)s = htv$. We infer that $s = h(tv + us)$. By hypothesis, $hR + aR \neq R$.

As $rR + aR = R$, we can find some $r', a' \in R$ such that $rr' + aa' = 1$. Since $aR + bR + cR = R$, we have some $x, y, z \in R$ such that $ax + by + cz = 1$. Hence, $ax + b(rr' + aa')y + c(rr' + aa')z = 1$, and so $a(x + ba'y + ca'z) + (br)(r'y) + (cr)r'z = 1$. This implies that $ax' + (br)y' + (c^n r)z' = 1$ for some $x', y', z' \in R$. It follows that $a(x' - y') + (a + br)y' + (c^n r)z' = 1$, and so $aR + (a + br)R + (c^n r)R = R$. This implies that $aR + hR = R$, a contradiction.

Therefore, $(a + br)R + c^n rR = R$, and so $(a + br)R + crR = R$. Then for the matrix A we have $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} a + br & cr \\ b & c \end{pmatrix} = B$. It suffices to prove B admits a diagonal reduction. Write $(a + br)x + cry = 1$ for some $x, y \in R$. Then the matrix $\begin{pmatrix} x & -cr \\ y & a + br \end{pmatrix}$ is

invertible, and we see that

$$\begin{pmatrix} 1 & 0 \\ -(bx+cy) & 1 \end{pmatrix} B \begin{pmatrix} x & -cr \\ y & a+br \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & ca \end{pmatrix},$$

as desired. Now we have,

$$\begin{pmatrix} 1 & 0 \\ -(bx+cy) & 1 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} x & -cr \\ y & a+br \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & ca \end{pmatrix}.$$

As $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -(bx+cy) & 1 \end{pmatrix}$ are invertible, then the multiplication of them is invertible. Hence A admits a diagonal reduction.

If a is π -adequate to c , then by similar way we have $arR + (br+c)R = R$. Write $arx + (br+c)y = 1$. In this case we have

$$\begin{pmatrix} x & y \\ -(br+c) & ar \end{pmatrix} A \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -(xa+yb) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -ac \end{pmatrix},$$

as required. \square

Corollary 4.9. *Every π -adequate ring is an elementary divisor ring.*

Proof. Let R be a π -adequate ring. In view of Theorem 3.8, R is J -stable, and so it has stable range 2 by Proposition 4.1. It follows from [7, Theorem 3.4] that R is Hermite. Thus, R is quasi adequate. Therefore we complete the proof, in terms of Theorem 4.8. \square

Furthermore, we can extend [17, Theorem 6] as follows.

Corollary 4.10. *Every quasi adequate ring has stable range 2.*

Proof. Let R be a quasi adequate ring. Then R is an elementary divisor ring, by Theorem 4.8. Hence, R is a Hermite ring. Therefore the proof is true, by [7, Theorem 3.4]. \square

A ring R is strongly completeable provided that $a_1R + \cdots + a_nR = dR$, $a_i, d \in R, i = 2, \dots, n$, implies there is a matrix over R with first row a_1, \dots, a_n and $\det(A) = d$. One easily checks that a Bézout ring is strongly completeable if and only if for any $d \in R, d \in a_1R + \cdots + a_nR, a_i \in R, i = 2, \dots, n$, implies there is a matrix over R with first row a_1, \dots, a_n and $\det(A) = d$.

Theorem 4.11. *Every J -stable ring is strongly completeable.*

Proof. Let R be a J -stable ring. Suppose that $a_1R + \cdots + a_nR = dR, a_i, d \in R, i = 2, \dots, n$. If $n = 2$, $d = a_1x_1 + a_2x_2$ for some $x_1, x_2 \in R$. Then $-x_2, x_1$ works as a second row. Suppose that the assertion holds for $k < n (n \geq 3)$. Write $d = a_1x_1 + \cdots + a_nx_n, a_1 = dq_1, \dots, a_n = dq_n$ for some $x_1, \dots, x_n, q_1, \dots, q_n \in R$. Let $c = x_1q_1 + \cdots + x_nq_n - 1$. Then $dc = 0$. Further, we have $q_1R + \cdots + q_{n-1}R + (q_nx_n - c)R = R$.

Case I. $q_1R + \cdots + q_{(n-2)}R \not\subseteq J(R)$. In view of Proposition 2.4, we can find some $z \in R$ such that $q_1R + \cdots + q_{(n-2)}R + (q_{(n-1)} + (q_nx_n - c)z)R = R$. Hence, $d \in dq_1R + \cdots + dq_{(n-2)}R + (dq_{(n-1)} + d(q_nx_n - c)z)R = a_1R + \cdots + a_{(n-2)}R + (a_{(n-1)} + a_nx_nz)R$. We infer that $dR = a_1R + \cdots + a_{(n-2)}R + (a_{(n-1)} + a_nx_nz)R$. By hypothesis, there exists an $(n-1) \times (n-1)$ matrix D whose first row is $a_1, \dots, a_{n-2}, a_{n-1} + a_nx_nz$ and $\det(D) = d$.

Let

$$A = \begin{pmatrix} & & & a_n \\ & & & 0 \\ & D & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n-2} & \mathbf{0} & \\ \mathbf{0} & 1 & 0 \\ & -x_n z & 1 \end{pmatrix}.$$

Then A is the required matrix.

Case II. $q_1 R + \cdots + q_{n-2} R \subseteq J(R)$. Then q_{n-1} or $q_n x_n - c$ is not in $J(R)$. Suppose that $q_{n-1} \notin J(R)$. Then $(q_1 + q_{n-1})R + q_2 R + \cdots + q_{n-2} R \not\subseteq J(R)$. Clearly, $(q_1 + q_{n-1})R + q_2 R + \cdots + q_{n-2} R + q_{n-1} R + (q_n x_n - c)R = R$. Similarly to Case I, we have a $z \in R$ such that $(q_1 + q_{n-1})R + q_2 R + \cdots + q_{n-2} R + (q_{n-1} + q_n(x_n z - c))R = R$. Thus, $d \in d(q_1 + q_{n-1})R + \cdots + dq_{n-2} R + (dq_{n-1} + dq_n x_n z)R = (a_1 + a_{n-1})R + \cdots + a_{(n-2)}R + (a_{(n-1)} + a_n x_n z)R$. It follows that $dR = (a_1 + a_{n-1})R + \cdots + a_{n-2}R + (a_{n-1} + a_n x_n z)R$. By the discussion in Case I, we can find a matrix A whose first row is $a_1 + a_{n-1}, a_2, \dots, a_{n-1}, a_n$ and $\det(A) = d$. Let

$$B = A \begin{pmatrix} I_{n-2} & \mathbf{0} \\ -1 & 0 & \\ 0 & 0 & I_2 \end{pmatrix}.$$

Then B is the desired matrix. Suppose that $q_n x_n - c \notin J(R)$. Then $(q_1 + (q_n x_n - c))R + q_2 R + \cdots + q_{n-2} R \not\subseteq J(R)$. Similarly, we prove that there exists a matrix whose first row is a_1, \dots, a_n and its determinant is d .

By induction, the theorem is proved. \square

Corollary 4.12. Every J-adequate ring and every π -adequate ring are strongly completable.

Immediately, we see that every adequate ring is strongly completable. Thus, every regular ring is strongly completable.

Example 4.13. For any $n \in \mathbb{N}$, \mathbb{Z}_n is strongly completable. As every homomorphic image of a principal ideal ring is principal ideal and \mathbb{Z}_n is a homomorphic image of \mathbb{Z} , \mathbb{Z}_n is a principal ideal ring, and then a Bézout ring. One easily checks that every nonzero prime ideal of \mathbb{Z}_n is a maximal ideal, and so every prime ideal of \mathbb{Z}_n is contained in a unique maximal ideal. As it is a finite ring, for every element s of \mathbb{Z}_n , we see that $Z(s)$, i.e., the set of maximal ideals containing s , is finite. In light of [4, Theorem 4.3], \mathbb{Z}_n is adequate, and we are through.

We note that Theorem 4.11 extend [9, Theorem 2.1] as well. Also every ring having almost stable range 1 is strongly completable. As a consequence, we have

Corollary 4.14 [8, Theorem]. Every Dedekind domain is strongly completable.

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